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A method of integration over matrix variables: II

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Abstract. A method for evaluating the integral

$$Z_p(g, c) = \int \prod_1^p \mathrm{d}M_i \exp\left(-\sum_1^p \operatorname{tr}\left(M_i^2 + \frac{g}{n} M_i^4\right) + 2c \sum_1^{p-1} \operatorname{tr} M_i M_{i+1} \right)$$

over p $n \times n$ Hermitian matrices is given in the limit of large n. This is an adaptation of an earlier article and should be read in conjunction with it. Explicit equations are written only for the case of three matrices.

1. Introduction

Consider the integral over a chain of matrices

$$Z_p(g, c) = \int \exp\left(-\sum_{i=1}^p V(M^{(i)}) + 2c \sum_{i=1}^{p-1} \operatorname{tr} M^{(i)} M^{(i+1)} \right) \prod_{i=1}^p \mathrm{d}M^{(i)}, \qquad (1.1)$$

where $M^{(1)}, M^{(2)}, \ldots, M^{(p)}$ are $n \times n$ Hermitian matrices,

$$V(M) = \operatorname{tr} M^2 + (g/n) \operatorname{tr} M^4, \qquad (1.2)$$

$$\mathrm{d}M = \prod_{i=1}^n \mathrm{d}M_{ii} \prod_{1 \leq i < j \leq n} \mathrm{d}(\operatorname{Re} M_{ij})\, \mathrm{d}(\operatorname{Im} M_{ij}), \qquad (1.3)$$

and all integrals run from $-\infty$ to ∞. We are interested in the limit

$$\lim_{n\to\infty} \frac{1}{n^2} \ln \frac{Z_p(g, c)}{Z_p(0, c)} = -E_0(p, g, c). \qquad (1.4)$$

The special cases $p = 1$ and 2 of equation (1.1) have been considered before (Brézin *et al* 1978, Itzykson and Zuber 1980, Bessis 1979, Mehta 1980). The physical motivation for the consideration of such integrals is described in the first two references above, where they are shown to be relevant to the problem of counting planar vacuum diagrams in a ϕ^4 theory. For the evaluation of the limit (1.4) in the general case, we give below an adaptation of the method described earlier in Mehta 1979. Thus $E_0(p, g, c)$ can in principle be evaluated for any p. However, the algebra becomes tedious as p increases and soon becomes prohibitive. Only the result for $p = 3$ is given explicitly.

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2. Integration over angle variables

The integration over angle variables of the matrices $M^{(1)}, M^{(2)}, \ldots$, in (1.1) above can be done step by step by using equation (2.24) of Mehta 1979 (denoted by ML in the following)

$$\int \mathrm{d}B \exp(-W(B)+2c \operatorname{tr} AB) = \left(\frac{\pi}{2c}\right)^{(1/2)n(n-1)} \int \mathrm{d}Y \exp\left(-W(Y)+2c\sum_i x_i y_i\right)\frac{\Delta(Y)}{\Delta(X)}, \tag{2.1}$$

which is valid for any function $W(B)$ depending only on the eigenvalues of B. Here $X \equiv \{x_i; i=1,2,\ldots,n\}$ are the eigenvalues of A, $Y=\{y_i\}$ are those of B, $\Delta(X) = \Pi_{i<j}(x_i - x_j)$ and similarly for $\Delta(Y)$. Thus

$$Z_p(g,c) = K\int \exp\left(-\sum_{i=1}^{p} V(X^{(i)}) + 2c\sum_{i=1}^{p-1}\sum_{j=1}^{n} x_j^{(i)} x_j^{(i+1)}\right)\Delta(X^{(1)})\,\Delta(X^{(p)})\prod_{i=1}^{p} \mathrm{d}X^{(i)}, \tag{2.2}$$

where $X^{(i)} \equiv \{x_j^{(i)}; j=1,2,\ldots,n\}$ are the eigenvalues of $M^{(i)}$. The constant K in front of the integral depends on c, n and p and can be determined by setting $V(X)=\Sigma_i x_i^2$. Its value is (see § 5.1 below),

$$K = \left(\frac{\pi^p}{(2c)^{p-1}}\right)^{n(n-1)/2}\left(\prod_1^n i!\right)^{-1} \tag{2.3}$$

3. Integration over eigenvalues: orthogonal polynomials

As in ML, we write

$$\Delta(X) = \det\{P_{i-1}(x_j)\}_{i,j=1,2,\ldots,n}, \tag{3.1}$$

where $P_i(x)$ is a polynomial with x^i as the highest order term,

$$P_i(x) = x^i + \text{lower powers of } x. \tag{3.2}$$

We shall choose these polynomials such that

$$\int P_i(x_1)P_j(x_p)\exp\left(-\sum_{k=1}^{p} V(x_k) + 2c\sum_{k=1}^{p-1} x_k x_{k+1}\right)\mathrm{d}x_1\ldots\mathrm{d}x_p = h_i\delta_{ij}. \tag{3.3}$$

Therefore, as in ML,

$$Z_p(g,c) = Kn!\prod_0^{n-1} h_i(g,c). \tag{3.4}$$

Since the exponential weight factor is not altered by a simultaneous change of sign of all its variables, one sees that

$$P_i(-x) = (-1)^i P_i(x). \tag{3.5}$$

As in ML let

$$\begin{aligned} xP_i(x) &= P_{i+1}(x) + R_i P_{i-1}(x) + S_i P_{i-3}(x) + \ldots \\ &= \sum_j \alpha_{ij} P_j(x), \end{aligned} \tag{3.6}$$

and

$$P_i'(x) \equiv \frac{\mathrm{d}}{\mathrm{d}x} P_i(x) = iP_{i-1}(x) + \epsilon_i P_{i-3}(x) + \dots$$
$$= \sum_j \beta_{ij} P_j(x). \tag{3.7}$$

Differentiating both sides of (3.6) and expressing everything as linear combinations of $P_j(x)$, we obtain

$$[\alpha, \beta] \equiv \alpha\beta - \beta\alpha = \mathbb{1}, \tag{3.8}$$

where $\mathbb{1}$ is the unit matrix (of infinite dimension). This equation completely determines the matrix β. An explicit expression of the solution as a series in terms of α is given in the Appendix.

To obtain the relations between the h_i and R_i, $S_i, \dots$ in a convenient form, we introduce the notations

$$v(x, y) = V(x) - 2c\,xy = x^2 + (g/n)x^4 - 2cxy, \tag{3.9}$$

$$(\mathcal{M}f)(x) = \int \mathrm{d}y \exp(-v(y, x))f(y), \tag{3.10}$$

$$\mathcal{M}^q f = \mathcal{M}(\mathcal{M}^{q-1}f), \qquad q > 1 \tag{3.11}$$

$$\langle f, g\rangle = \int \mathrm{d}x \exp(-V(x))f(x)g(x). \tag{3.12}$$

Then

$$\langle \mathcal{M}f, g\rangle = \langle f, \mathcal{M}g\rangle, \tag{3.13}$$

and the orthogonality property (3.3) can be written as

$$h_i\delta_{ij} = \langle P_i, \mathcal{M}^{p-1}P_j\rangle = \langle \mathcal{M}^{q-1}P_i, \mathcal{M}^{p-q}P_j\rangle, \qquad 1 \leq q \leq p. \tag{3.14}$$

Integration by parts gives

$$\int \left(y + \frac{2g}{n}y^3 - cx\right) \exp(-v(y, x))P_i(y)\,\mathrm{d}y \equiv -\frac{1}{2}\int P_i(y)\frac{\mathrm{d}}{\mathrm{d}y}\exp(-v(y, x))\,\mathrm{d}y$$
$$= \frac{1}{2}\int \exp(-v(y, x))P_i'(y)\,\mathrm{d}y = \frac{1}{2}\sum_j \beta_{ij}(\mathcal{M}P_j)(x). \tag{3.15}$$

The left-hand side can also be evaluated by expressing $yP_i(y)$ and $y^3P_i(y)$ as linear combinations of $P_j(y)$. Thus writing

$$x(\mathcal{M}P_i)(x) = \sum_j \alpha_{(1);ij}(\mathcal{M}P_j)(x), \tag{3.16}$$

we obtain

$$\alpha_{(1)} = \frac{1}{c}\left(\alpha + \frac{2g}{n}\alpha^3\right) - \frac{1}{2c}\beta. \tag{3.17}$$

Similarly, integrating

$$(\mathcal{M}^{q-1}P_i)(y)(\mathrm{d}/\mathrm{d}y)\exp(-v(y, x)) \tag{3.18}$$

in two different ways, one obtains

$$\alpha_{(q)} = \frac{1}{c}\left(\alpha_{(q-1)} + \frac{2g}{n}\alpha^3_{(q-1)}\right) - \alpha_{(q-2)}, \qquad q \geq 2, \tag{3.19}$$

where the matrix $\alpha_{(q)}$ is defined by

$$x(\mathcal{M}^q P_i)(x) = \sum_j \alpha_{(q);ij}(\mathcal{M}^q P_j)(x). \tag{3.20}$$

Thus the matrices $\alpha_{(1)}, \alpha_{(2)}, \ldots, \alpha_{(p-1)}$ can be successively expressed in terms of $\alpha_{(0)} \equiv \alpha$ and β, that is in terms of α.

Now use equations (3.20) and (3.14) in the identity

$$\langle x\mathcal{M}^{q-1}P_i, \mathcal{M}^{p-q}P_j\rangle = \langle \mathcal{M}^{q-1}P_i, x\mathcal{M}^{p-q}P_j\rangle \tag{3.21}$$

to obtain

$$\alpha_{(q-1);ij}h_j = \alpha_{(p-q);ji}h_i. \tag{3.22}$$

The equations (3.17), (3.19) and (3.22) together with (3.6) and (3.7) determine all the h_i, R_i, $S_i, \ldots$ in terms of h_0.

4. The large-n limit

Setting $f_i = h_i/h_{i-1}$, one obtains

$$\frac{1}{n^2}\ln\left(\frac{Z_p(g, c)}{Z_p(0, c)}\right) = \frac{1}{n}\ln\left(\frac{h_0(g, c)}{h_0(0, c)}\right) + \frac{1}{n}\sum_{i=1}^{n}\left(1 - \frac{i}{n}\right)\ln\left(\frac{f_i(g, c)}{f_i(0, c)}\right). \tag{4.1}$$

As

$$\frac{h_0(g, c)}{h_0(0, c)} = 1 + O\left(\frac{1}{n}\right), \tag{4.2}$$

and $f_i(g, c)/f_i(0, c)$ is well-behaved near $i = 1$ and $i = n$, we obtain in the large-n limit

$$-E_0(p, g, c) = \lim_{n\to\infty}\frac{1}{n^2}\ln\frac{Z_p(g, c)}{Z_p(0, c)} = \int_0^1 (1 - x)\ln\frac{f(x)}{f_0(x)}\,dx, \tag{4.3}$$

where

$$f_i \sim nf(x), \qquad x = i/n, \tag{4.4}$$

and $f_0(x)$ denotes the value of $f(x)$ when $g = 0$.

5. Some particular cases

To be a little familiar with the method let us examine a few particular examples.

5.1. The case $g = 0$

Equations (3.17) and (3.19) read

$$\alpha_{(1)} = (1/c)(\alpha - \tfrac{1}{2}\beta), \tag{5.1}$$

$$\alpha_{(q)} = (1/c)\alpha_{(q-1)} - \alpha_{(q-2)}, \qquad q \geq 2. \tag{5.2}$$

Their solution is

$$\alpha_{(q)} = c^{-q}(D_q\alpha - \tfrac{1}{2}D_{q-1}\beta), \qquad q \geq 0, \tag{5.3}$$

where the sequence of numbers D_q is determined by

$$D_{-1} = 0, \qquad D_0 = D_1 = 1, \tag{5.4}$$

$$D_q = D_{q-1} - c^2 D_{q-2}, \qquad q \geq 2, \tag{5.5}$$

i.e.

$$D_q = (1-4c^2)^{-1/2}\left[\left(\frac{1+(1-4c^2)^{1/2}}{2}\right)^{q+1} - \left(\frac{1-(1-4c^2)^{1/2}}{2}\right)^{q+1}\right] \tag{5.6}$$

Thus the matrices $\alpha_{(q)}$ are explicitly known. For example,

$$\alpha_{(q);j-1,j} = c^{-q}D_q, \tag{5.7}$$

$$\alpha_{(q);j,j-1} = c^{-q}(D_qR_j - \tfrac{1}{2}jD_{q-1}). \tag{5.8}$$

Equation (3.22) for $i = j-1$ and $q = p$ gives

$$c^{-p+1}D_{p-1}h_j = R_jh_{j-1} \tag{5.9}$$

while for $q = p-1$ it gives

$$c^{-p+3}D_{p-2}h_j = (R_j - j/2)h_{j-1}. \tag{5.10}$$

Elimination of R_j from the last two equations gives

$$f_j = h_j/h_{j-1} = \tfrac{1}{2}jc^{p-1}(D_{p-1} - c^2D_{p-2})^{-1} = \tfrac{1}{2}jc^{p-1}D_p^{-1}, \tag{5.11}$$

or

$$h_j = 2^{-j}j!c^{j(p-1)}D_p^{-j}h_0. \tag{5.12}$$

It is not very difficult to evaluate h_0:

$$h_0 \equiv h_0(0,c) = \int \exp\left(-\sum_1^p x_j^2 + 2c\sum_1^{p-1} x_jx_{j+1}\right) \mathrm{d}x_1 \ldots \mathrm{d}x_p = \pi^{p/2}D_p^{-1/2}. \tag{5.13}$$

This gives finally

$$\begin{aligned} Z_p(0,c) &= K\,.\,n!\prod_0^{n-1} h_j(0,c) \\ &= K\,.\,n!\prod_0^{n-1}\{2^{-j}j!c^{j(p-1)}D_p^{-j-1/2}\pi^{p/2}\} \\ &= K2^{-n(n-1)/2}c^{n(n-1)(p-1)/2}D_p^{-n^2/2} + np/2\prod_1^n j! \end{aligned} \tag{5.14}$$

One can calculate $Z_p(0,c)$ directly from equation (1.1) as well,

$$Z_p(0,c) = 2^{-(p/2)n(n-1)}h_0^{n^2} = 2^{-pn(n-1)/2}\pi^{pn^2/2}D_p^{-n^2/2}. \tag{5.15}$$

Equating the last two expressions one obtains the value of K given in equation (2.3).

5.2. The case $p=2$

Here

$$\alpha_{(1)}=(1/c)[\alpha+(2g/n)\alpha^3-\tfrac{1}{2}\beta], \tag{5.16}$$

and equation (3.22) gives ($q=1$),

$$c\alpha_{ij}h_j=[\alpha+(2g/n)\alpha^3-\tfrac{1}{2}\beta]_{ji}h_i. \tag{5.17}$$

Setting $j=i+1$, $i-1$ and $i-3$, one obtains the equations (3.16), (3.18) and (3.20) of ML.

5.3. The case $p=3$

Here in addition to (5.16) we have

$$\alpha_{(2)}=(1/c)[\alpha_{(1)}+(2g/n)\alpha^3_{(1)}]-\alpha. \tag{5.18}$$

Equation (3.22) for $q=2$ states that

$$\alpha_{(1);ij}h_j=\alpha_{(1);ji}h_i. \tag{5.19}$$

In particular

$$\alpha_{(1);ij}=0 \qquad \text{for } |i-j|>3, \tag{5.20}$$

$$\alpha_{(1);i,i+3}h_{i+3}=\alpha_{(1);i+3,i}h_i \tag{5.21}$$

and

$$\alpha_{(1);i,i+1}h_{i+1}=\alpha_{(1);i+1,i}h_i. \tag{5.22}$$

Writing explicitly these matrix elements from (5.16), (3.6) and (3.7) we obtain

$$c\alpha_{(1);i+3,i}=\frac{2g}{n}\frac{h_{i+3}}{h_i}, \tag{5.23}$$

and

$$c\alpha_{(1);i,i-1}=R_i\left(1+\frac{2g}{n}(R_{i-1}+R_i+R_{i+1})\right)+\frac{2g}{n}(S_i+S_{i+1}+S_{i+2})-\frac{i}{2} \tag{5.24}$$

$$=\frac{h_i}{h_{i-1}}c\alpha_{(1);i-1,i}$$

$$=\frac{h_i}{h_{i-1}}\left(1+\frac{2g}{n}(R_{i-1}+R_i+R_{i+1})\right). \tag{5.25}$$

Using equations (5.20) and (5.23)–(5.25) in equation (5.18), one can write the equations (3.22) for $q=3$. As such they are cumbersome and therefore useless. However, for the leading term in equation (1.4), one can replace the f_i, R_i, S_i, ... by continuous functions

$$f_i\sim nf(x), \qquad R_i\sim nR(x), \qquad S_i\sim n^2S(x), \qquad i\sim nx. \tag{5.26}$$

The resulting equations are simple. Equations (5.23)–(5.25) are rewritten as

$$(c/n^2)\alpha_{(1);i+3,i}=2gf^3, \tag{5.27}$$

$$(c/n)\alpha_{(1);i+1,i}=c\alpha_{(1);i,i+1}f=R+6g(S+R^2)-(x/2)=f(1+6gR), \tag{5.28}$$

while equation (3.22) for $q=3$ gives

$$6gf^2(1+6gR)[(1+6gR)^2+2gf(1+6gR)+8g^2f^2]+c^2f(1+6gR)=c^4(R+f), \tag{5.29}$$

and

$$2gf^3\{c^2+(1+6gR)^3+12gf(1+6gR)^2+24g^3f^3\}=c^4S. \tag{5.30}$$

Equations (5.28)–(5.30) determine f, R and S. One can eliminate S quite easily. To eliminate R is a little lengthy, though it presents no difficulties.

For still higher values of $p\,(p\geqslant 4)$ equation (3.22) contains the necessary information to obtain, at least implicitly, the leading term as $n\to\infty$ of the integral (1.1). But, as is already evident from the case $p=3$, the algebra becomes progressively more tedious.

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Appendix: Solution of equation (3.8)

The matrices considered here are all of infinite order; for example, M will have matrix elements M_{ij} with $i, j=1, 2, 3, \ldots$.

First consider the matrix equation

$$[A, X]=B, \tag{A1}$$

with unknown X, given A,

$$A_{ij}=\begin{cases}1 & \text{if } j=i+1\\ 0 & \text{otherwise,}\end{cases} \tag{A2}$$

and a known matrix B. The general solution of equation (A1) is the sum of that of the homogeneous equation

$$[A, X^0]=0, \tag{A3}$$

and a particular solution of (A1). Writing the matrix elements of the homogeneous equation (A.3), one sees that

$$X^0_{j1}=0, j\geqslant 2; \qquad X^0_{i+1,k}=X^0_{i,k-1}, k\geqslant 2;$$

while X^0_{1j} is arbitrary for $j\geqslant 1$. In other words

$$X^0=\sum_{k=0}^{\infty} x_k A^k \tag{A4}$$

with arbitrary constants x_k.

Using $AA^{\mathrm{T}}=\mathbb{1}$, where A^T denotes the transpose of A and $\mathbb{1}$ the unit matrix, one can verify that

$$\mathscr{A}(B)=\sum_{k=1}^{\infty}(A^{\mathrm{T}})^k B A^{k-1} \tag{A5}$$

is a particular solution of equation (A1). Note that the matrix element (i, j) on the right hand side of (A5) has only a finite number of non-zero terms for i and j finite.

Therefore the general solution of (A1) is the sum of (A4) and (A5).

Note that $\mathscr{A}$ is linear,

$$\mathscr{A}(B_1+B_2)=\mathscr{A}(B_1)+\mathscr{A}(B_2). \tag{A6}$$

To express β as a series in α, we write

$$\alpha=A+\bar{\alpha}, \qquad \beta=\mathscr{A}(\mathbb{1})+\bar{\beta}. \tag{A7}$$

Then equation (3.8) can be written as

$$[A, \bar{\beta}]=[\mathscr{A}(\mathbb{1}), \bar{\alpha}]+[\bar{\beta}, \bar{\alpha}], \tag{A8}$$

so that from the foregoing considerations

$$\bar{\beta}=\sum_{k=0}^{\infty} x_k A^k+\mathscr{A}([\mathscr{A}(\mathbb{1}), \bar{\alpha}])+\mathscr{A}([\bar{\beta}, \bar{\alpha}]).$$

But $\beta_{ij}=0=\bar{\beta}_{ij}$ for $i \leqslant j$. Hence

$$\bar{\beta}=\mathscr{A}([\mathscr{A}(\mathbb{1}), \bar{\alpha}])+\mathscr{A}([\bar{\beta}, \bar{\alpha}]). \tag{A9}$$

Iterating this last equation several times we obtain

$$\bar{\beta}=\mathscr{A}([\mathscr{A}(\mathbb{1}), \bar{\alpha}])+\mathscr{A}\{[\mathscr{A}([\mathscr{A}(\mathbb{1}), \bar{\alpha}]), \bar{\alpha}]\}+\ldots, \tag{A.10}$$

or introducing the notation

$$\mathscr{A}_0(\bar{\alpha})=\mathscr{A}(\mathbb{1}), \qquad \mathscr{A}_{k+1}(\bar{\alpha})=\mathscr{A}([\mathscr{A}_k(\bar{\alpha}), \bar{\alpha}]),\ k \geqslant 0, \tag{A11}$$

one can write

$$\beta=\sum_{k=0}^{\infty} \mathscr{A}_k(\bar{\alpha}). \tag{A12}$$

Let us say that the matrix M has the type m if $M_{ij}=0$ for $i-j<m$ and $M_{ij} \neq 0$ for some $i-j=m$. Thus the type of A is -1, while that of A^{T}, $\bar{\alpha}$ and β is $+1$. One can readily verify that the following statements are true:

(i) If the type of M_1 is m_1 and that of M_2 is m_2, then the type of $M_1 \pm M_2$ is $\geqslant$ min (m_1, m_2) and that of M_1M_2 is $\geqslant m_1+m_2$.

(ii) If the type of M is m, then that of $\mathscr{A}(M)$ is $m+1$.

(iii) The type of $\mathscr{A}(\mathbb{1})$ is 1.

(iv) If the type of $\mathscr{A}_k(\bar{\alpha})$ is $t(k)$, then that of $[\mathscr{A}_k(\bar{\alpha}), \bar{\alpha}]$ is $\geqslant t(k)+1$ and therefore that of $\mathscr{A}_{k+1}(\bar{\alpha})$ is $\geqslant t(k)+2$; in other words $t(k+1) \geqslant t(k)+2$.

(v) The type of $\mathscr{A}_k(\bar{\alpha})$ *is* $\geqslant 2k+1$.

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